

## Further Properties of the Scale Function on a Totally Disconnected Group

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A new characterisation of the scale function on the locally compact group  $G$  is given. It is shown that for  $x$  in  $G$  the scale of  $x$ ,  $s(x)$ , is the minimum value attained by the index  $[xUx^{-1} : xUx^{-1} \cap U]$  as  $U$  ranges over all compact open subgroups of  $G$ . The properties of the scale function when passing to subgroups and quotient groups of  $G$  and under increasing unions of groups are also described. © 2001 Academic Press

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### 1. INTRODUCTION

There is a function, called the scale function and denoted by  $s$ , defined on each totally disconnected locally compact group  $G$ . The scale function is a continuous map from  $G$  to the positive integers and has the following properties.

S1.  $s(x) = 1 = s(x^{-1})$  if and only if there is a compact open subgroup  $U$  of  $G$  such that  $xUx^{-1} = U$ .

S2.  $s(x^n) = s(x)^n$  for every positive integer  $n$  and every  $x$  in  $G$ .

S3.  $\Delta(x) = s(x)/s(x^{-1})$  for every  $x$  in  $G$ , where  $\Delta: G \rightarrow \mathbb{R}^+$  denotes the modular function.

The scale function is defined in [6]. Its existence is closely related to the structure of totally disconnected groups. For instance, in [4] the scale

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function and the associated notion of tidy subgroup are used to prove a conjecture of Hofmann and Mukherjea made in [3]. In [7] the scale function alone is used to prove another conjecture of Hofmann.

In this paper the relation between the scale function on  $G$  and those on closed subgroups, closed normal subgroups, and quotient groups of  $G$  is investigated. Property S3 suggests that, when passing to a subgroup or quotient group, the scale function may factor in the same way as the modular function does; see [2, (15.23)]. The theorems and examples given in Sections 4 and 6 show the extent to which this is so. In Section 5 it is shown how the scale function behaves on increasing unions and inverse limits of groups. Proposition 5.3 and Example 6.6 are the results referred to in [5]. The results in these two sections extend work of Herman in [1] and are a first step towards developing functorial properties of the scale function.

The scale function arises in [6] from some structure theorems for totally disconnected groups proved there. A new and more direct characterisation is given in Theorem 3.1. For the proof of this characterisation it is necessary to recapitulate some of the arguments from [6] and to give a revised, and much clearer, proof of the last step. We begin by recalling, and extending slightly, some of the concepts and theorems from [6].

## 2. TIDY SUBGROUPS AND THE SCALE FUNCTION

The value of the scale function at an element  $x$  is defined in terms of the inner automorphism  $\alpha_x: h \mapsto xhx^{-1}$  of conjugation by  $x$ . This definition extends to arbitrary continuous automorphisms  $\alpha$  of  $G$ , as may easily be seen by examining the proofs in [6] or by applying the original definition to the semidirect product  $G \rtimes_{\alpha} \mathbb{Z}$ .

**DEFINITION 2.1.** Let  $G$  be a totally disconnected locally compact group and  $\alpha$  be a continuous automorphism of  $G$ . Let  $U$  be a compact open subgroup of  $G$  and set

$$U_+ = \bigcap_{n=0}^{\infty} \alpha^n(U) \quad \text{and} \quad U_- = \bigcap_{n=0}^{\infty} \alpha^{-n}(U).$$

Then  $U$  is said to be *tidy* for  $\alpha$  if it satisfies:

- T1.  $U = U_+ U_- = U_- U_+$ , and
- T2.  $\bigcup_{n=0}^{\infty} \alpha^n(U_+)$  and  $\bigcup_{n=0}^{\infty} \alpha^{-n}(U_-)$  are closed in  $G$ .

It is shown in [6] that tidy subgroups exist for every continuous automorphism  $\alpha$  of  $G$ . Note that  $\alpha(U)$  is a compact group and  $\alpha(U) \cap U$  is open.

Hence the index  $[\alpha(U) : \alpha(U) \cap U]$  is finite. It is further shown in [6] that this index is independent of the choice of subgroup tidy for  $\alpha$ . This index is thus a well defined function on  $\text{Aut}(G)$ .

DEFINITION 2.2. The *scale function*,  $s_G : \text{Aut}(G) \rightarrow \mathbb{N}$ , is defined by

$$s_G(\alpha) = [\alpha(U) : \alpha(U) \cap U] \quad (\alpha \in \text{Aut}(G)),$$

where  $U$  is a subgroup tidy for  $\alpha$ .

The subscript  $G$  on the scale function was not used in [6] because only one group at a time was considered there but here we shall be comparing the scale functions of different groups.

### 3. THE SCALE OF $\alpha$ AS A MINIMUM VALUE

In this section alternative characterisations of the scale function and tidy subgroups are proved. For this it is necessary to recapitulate the proof of the existence of subgroups satisfying T1 and T2. In outline, the proof starts with an arbitrary compact open subgroup  $U$  of  $G$  and constructs a subgroup tidy for  $\alpha$  in three steps.

*Step 1.* Find an open subgroup  $V$  of  $U$  which satisfies T1.

*Step 2.* Identify a certain compact subgroup  $L$  of  $G$  which satisfies  $\alpha(L) = L$ .

*Step 3.* Combine  $V$  and  $L$  to produce a compact open subgroup  $W$  which satisfies T1 and T2, i.e., is tidy for  $\alpha$ .

These steps are gone through in more detail in the proof of Theorem 3.1 below. In Steps 1 and 2 this involves recalling the main points from [6] but a new, and clearer, proof of Step 3 is given in full. We shall frequently make use of the fact that  $[\alpha(U) : \alpha(U) \cap U] = [\alpha(U_+) : U_+]$  for any  $U$  satisfying T1. This follows from [6, the proof of Lemma 1].

THEOREM 3.1. *Let  $G$  be a totally disconnected locally compact group. Then for every  $\alpha$  in  $\text{Aut}(G)$*

$$s_G(\alpha) = \min\{[\alpha(U) : \alpha(U) \cap U] : U \text{ is a compact open subgroup of } G\}.$$

*A compact open subgroup  $U$  of  $G$  is tidy for  $\alpha$  if and only if*

$$s_G(\alpha) = [\alpha(U) : \alpha(U) \cap U],$$

*i.e., if and only if the minimum is attained at  $U$ .*

*Proof.* Let  $U$  be an arbitrary compact open subgroup of  $G$ . We shall go through Steps 1, 2, and 3 and observe that

$$[\alpha(U) : \alpha(U) \cap U] \geq [\alpha(V) : \alpha(V) \cap V] \geq [\alpha(W) : \alpha(W) \cap W].$$

Since  $U$  is an arbitrary subgroup and  $W$  is tidy, this will prove the first part of the theorem. We shall also see that, if  $U$  is not already tidy, then one of the inequalities is strict, thus proving the second part.

*Step 1.* [6, Lemma 1] shows that there is an  $N$  such that  $V = \bigcap_{n=0}^N \alpha^n(U)$  satisfies T1. Then  $V_+ = U_+$ , where  $U_+$  and  $V_+$  are defined as in Definition 2.1.

Since  $\alpha(U) \supset \alpha(U_+)(\alpha(U) \cap U)$ , we have

$$[\alpha(U) : \alpha(U) \cap U] \geq [\alpha(U_+)(\alpha(U) \cap U) : \alpha(U) \cap U], \quad (1)$$

with equality if and only if  $\alpha(U) = \alpha(U_+)(\alpha(U) \cap U)$ . (Note that  $\alpha(U_+)(\alpha(U) \cap U)$  need not be a group but it is a set of cosets over  $\alpha(U) \cap U$  and the index on the right is interpreted to mean the number of these cosets.) It follows from [6, proof of Lemma 1] that this occurs if and only if  $U$  satisfies T1. Hence there is equality in (1) if and only if  $U$  already satisfies T1. Now for  $u$  and  $v$  in  $\alpha(U_+)$ ,  $u(\alpha(U) \cap U) = v(\alpha(U) \cap U)$  if and only if  $v^{-1}u \in \alpha(U_+) \cap U = U_+$ , whence

$$[\alpha(U_+)(\alpha(U) \cap U) : \alpha(U) \cap U] = [\alpha(U_+) : U_+] = [\alpha(V_+) : V_+].$$

Therefore

$$[\alpha(U) : \alpha(U) \cap U] \geq [\alpha(V_+) : V_+] = [\alpha(V) : \alpha(V) \cap V]$$

with equality if and only if  $U$  already satisfies T1.

*Step 2.* Define

$$\mathcal{L} = \{u \in G : \alpha^n(u) \in V \text{ for all but finitely many integers } n\} \quad (2)$$

and let  $L$  be the closure of  $\mathcal{L}$ . It is clear that  $\alpha(L) = L$  and it is shown in [6, Lemma 6] that  $L$  is compact.

[6, Corollary to Lemma 3] asserts that  $V$  is tidy if and only if  $\mathcal{L} = V_+ \cap V_-$ . Now  $V_+ \cap V_-$  is just  $\{u \in G : \alpha^n(u) \in V \text{ for every } n\}$  and so  $V$  is tidy if and only if  $\mathcal{L} = L \subset V$ . If  $V$  is not tidy, then a tidy subgroup  $W$  is obtained by “adding”  $L$  to  $V$  as described in the next step.

*Step 3.* The following criterion for a group element to belong to  $L$  will be cited repeatedly.

**LEMMA 3.2.** *Let  $u$  be in  $V_+$  and suppose that  $\{\alpha^n(u)\}_{n \geq 0}$  has an accumulation point. Then  $u$  belongs to  $L$ .*

*Proof.* Let  $c$  be an accumulation point of  $\{\alpha^n(u)\}_{n \geq 0}$  and let  $N$  be a positive integer. Choose  $n > N$  and  $m > 2n$  such that  $\alpha^m(u)$  and  $\alpha^n(u)$  belong to  $cV$ . Then  $\alpha^n(u)^{-1}\alpha^m(u)$  belongs to  $V$  and there are  $v \in V_+$  and  $w \in V_-$  such that  $\alpha^n(u)^{-1}\alpha^m(u) = vw$ . Put  $k_N = \alpha^{-m}(w) = \alpha^{-m}(v^{-1})\alpha^{n-m}(u)^{-1}u$ .

Then for  $p < N$  we have

$$\alpha^p(k_N u^{-1}) = \alpha^{p-m}(v^{-1})\alpha^{p+n-m}(u)^{-1} \quad (3)$$

which belongs to  $V_+$  because  $p - m < 0$  and  $v^{-1} \in V_+$  and  $p + n - m < 0$  and  $u \in V_+$ . The same reasoning implies that  $k_N \in V_+$  and we also have  $\alpha^m(k_N) = w \in V_-$ , so that  $k_N \in \mathcal{L} \subset L$ . Since  $L$  is compact, it follows that  $\{k_N\}_{N > 0}$  has an accumulation point,  $l$  say, in  $L$ . It follows from Eq. (3) that  $\alpha^p(lu^{-1}) \in V$  for every  $p \in \mathbb{Z}$ , so that  $lu^{-1} \in L$ . Therefore  $u \in L$ . ■

The obvious way to add  $L$  to  $V$  is to try to define  $W$  to be  $VL$ , but this set need not be a subgroup. The right definition of  $W$  uses instead

$$V' = \{v \in V : lvl^{-1} \in VL \text{ for all } l \in L\}. \quad (4)$$

LEMMA 3.3. (1)  $V'$  is an open subgroup of  $V$ .

(2) Each  $v \in V'$  satisfies  $lvl^{-1} \in V'L$  for all  $l \in L$ .

(3)  $V'L$  is a compact open subgroup of  $G$ .

*Proof.* (1) Let  $v_1$  and  $v_2$  be in  $V'$ . Then for every  $l \in L$  there are  $l_1 \in L$ ,  $u_1 \in V$  with  $lv_1 l^{-1} = u_1 l_1$  and  $l_2 \in L$ ,  $u_2 \in V$  with  $l_1 l_2 (l_1 l)^{-1} = u_2 l_2$ , so that

$$lv_1 v_2 l^{-1} = (lv_1 l^{-1})(lv_2 l^{-1}) = u_1(l_1 l_2 (l_1 l)^{-1})l_1 = u_1 u_2 l_2 l_1$$

belongs to  $VL$ . Hence  $V'$  is a semigroup. Since  $VL$  is closed,  $V'$  is a closed semigroup contained in the compact group  $V$ . It follows that  $V'$  is closed under taking inverses and is therefore a group.

Since  $V'$  contains the open set  $\bigcap_{l \in L} lVl^{-1}$ , it is an open subgroup of  $V$ .

(2) Let  $v$  be in  $V'$  and  $l$  be in  $L$ . Then there are  $v_1 \in V$  and  $l_1 \in L$  such that  $lv l^{-1} = v_1 l_1$ . Now  $v_1 = lv l^{-1} l_1^{-1}$  and so for each  $m \in L$  there are  $v_2 \in V$  and  $l_2 \in L$  such that

$$mv_1 m^{-1} = (ml)v(ml)^{-1}ml_1^{-1}m^{-1} = v_2 l_2 ml_1^{-1}m^{-1},$$

which belongs to  $VL$ . Hence  $v_1 \in V'$  and so in fact  $lvl^{-1}$  belongs to  $V'L$ .

(3) Since  $V'$  and  $L$  are compact sets,  $V'L$  is compact and, since  $V'$  is open,  $V'L$  is open. To see that it is a subgroup, let  $v_1 l_1$  and  $v_2 l_2$  be in

$V'L$ . Then there are  $v_3 \in V'$  and  $l_3 \in L$  such that

$$(v_1 l_1)(v_2 l_2) = v_1(l_1 v_2 l_1^{-1})l_1 l_2 = v_1 v_3 l_3 l_1 l_2.$$

Hence  $V'L$  is a subsemigroup of the group  $G$  and so, since it is compact, it is a subgroup of  $G$ . ■

We now define  $W$  to be  $V'L$ . There is an apparent asymmetry in this definition but, since  $V'L$ ,  $V'$ , and  $L$  are groups, we have  $V'L = (V'L)^{-1} = L^{-1}(V')^{-1} = LV'$ . Several lemmas are required to show that  $W$  is tidy.

LEMMA 3.4. *There is an integer  $p$  such that  $\alpha^{-p}(V_+) \subset V'_+$ .*

*Proof.* Since it is a subset of  $V$  and is invariant under  $\alpha$ ,  $V_+ \cap V_- \subset L$ . That  $V_+ \cap V_- \subset V'$  now follows from the definition of  $V'$ . Now  $\{\alpha^{-n}(V_+)\}_{n \geq 0}$  is a decreasing sequence of compact sets and  $\bigcap_{n \geq 0} \alpha^{-n}(V_+) = V_+ \cap V_-$ . Hence there is  $p > 0$  such that  $\alpha^{-p}(V_+)$  is contained in the open neighbourhood  $V'$  of  $V_+ \cap V_-$ . Since  $\{\alpha^{-n}(V_+)\}_{n \geq 0}$  is decreasing,  $\alpha^{-n}(V_+) \subset V'$  for every  $n \geq p$ , whence  $\alpha^{-p}(V_+) \subset V'_+$ . ■

LEMMA 3.5. (1)  $V'_+ = V' \cap V_+ = \{v \in V_+ : l v l^{-1} \in V_+ L \text{ for all } l \in L\}$ .

(2) Each  $v \in V'_+$  satisfies  $l v l^{-1} \in V'_+ L$  for all  $l \in L$ .

(3)  $V'_+ L$  is a compact group and  $V'_+ L = L V'_+$ .

*Proof.* (1) Since  $V' \subset V$  we have

$$V'_+ = \bigcap_{n \geq 0} \alpha^n(V') \subset V' \cap \left( \bigcap_{n \geq 0} \alpha^n(V) \right) = V' \cap V_+.$$

Now let  $v$  be in  $V' \cap V_+$ . Then, for each  $l \in L$ ,  $l v l^{-1} \in VL$  so that there are  $u \in V_+$ ,  $w \in V_-$ , and  $m \in L$  with  $l v l^{-1} = u w m$ . Now  $w \in V_-$  and  $w = u^{-1} l v l^{-1} m^{-1} \in V_+ L V_+ L$ , whence  $\{\alpha^{-n}(w)\}_{n \geq 0}$  has an accumulation point. Hence, by Lemma 3.2,  $w \in L$  and it follows that  $l v l^{-1} \in V_+ L$ . Therefore

$$V' \cap V_+ \subset \{v \in V_+ : l v l^{-1} \in V_+ L \text{ for all } l \in L\}.$$

Finally, let  $v$  belong to the latter set. Then for each  $n \geq 0$  and each  $l \in L$ ,

$$l \alpha^{-n}(v) l^{-1} = \alpha^{-n}(\alpha^n(l) v \alpha^n(l)^{-1}) \in \alpha^{-n}(V_+ L)$$

because  $\alpha^n(l) \in L$ . Since  $\alpha^{-n}(V_+ L) = \alpha^{-n}(V_+) L \subset VL$  for every  $n \geq 0$ , this shows that  $v \in V'_+$ . Thus

$$\{v \in V_+ : l v l^{-1} \in V_+ L \text{ for all } l \in L\} \subset V'_+$$

and the proof of (1) is complete.

Parts (2) and (3) are proved in exactly the same way as the corresponding parts in Lemma 3.3. ■

The corresponding statements hold for  $V'_-$  and  $V'_-L$ .

LEMMA 3.6.  $V' = V'_+V'_-$ .

*Proof.* Let  $v$  be in  $V'$ . Then  $v = uw$  for some  $u \in V_+$  and  $w \in V_-$ . We will show that  $u$  and  $w$  belong to  $V'$  which, by Lemma 3.5, will suffice to prove the desired factoring of  $V'$ . Since  $w = u^{-1}v$ , it suffices to show that  $u$  belongs to  $V'$ .

Let  $l$  be in  $L$ . By Lemma 3.4 there is  $p \geq 0$  such that  $\alpha^{-p}(u) \in V'_+$ . Since  $LV'_+ = V'_+L$  and since  $L$  is invariant under  $\alpha$ , it follows that

$$lul^{-1} = u_1m_1 \quad \text{for some } u_1 \in \alpha^p(V'_+) \quad \text{and} \quad m_1 \in L. \quad (5)$$

On the other hand,  $lul^{-1} \in VL$  and so there are  $u_2 \in V_+$ ,  $w_2 \in V_-$  and  $m \in L$  such that

$$lul^{-1} = u_2w_2m_2. \quad (6)$$

Now  $lul^{-1} = (lul^{-1})(lwl^{-1})$  and so (5) and (6) yield  $u_1m_1lwl^{-1} = u_2w_2m_2$  whence

$$u_2^{-1}u_1 = w_2m_2lw^{-1}l^{-1}m_1^{-1} \in V_-LV_-L.$$

It follows that  $\{\alpha^n(u_2^{-1}u_1)\}_{n \geq 0}$  has an accumulation point. Now  $u_2^{-1}u_1 \in \alpha^p(V_+)$  and so, by Lemma 3.2,  $u_2^{-1}u_1 \in L$ . Hence  $u_1 \in u_2L$  where  $u_2 \in V_+$  and so, by (5), it follows that  $lul^{-1} \in V_+L$ . Since this holds for every  $l \in L$  we have that  $u \in V'$ . ■

The next result shows that  $W$  satisfies T1.

LEMMA 3.7. (1)  $(V'L)_\pm = V'_\pm L$ .

(2)  $V'L = (V'L)_+(V'L)_-$ .

*Proof.* For each  $n \geq 0$  we have  $\alpha^{-n}(V'_+L) = \alpha^{-n}(V'_+)L \subset V'L$ . It follows that  $V'_+L \subset (V'L)_+$  and it may be shown similarly that  $V'_-L \subset (V'L)_-$ . Then both (1) and (2) are implied by the inclusions

$$V'L \subset V'_+V'_-L \subset (V'_+L)(V'_-L) \subset (V'L)_+(V'L)_- \subset V'L,$$

where the first inclusion follows from Lemma 3.6, the second is trivial, the third follows from above, and the fourth is trivial. ■

Define

$$\mathcal{X} = \{v \in G : \alpha^n(v) \in W \text{ for all but finitely many integers } n\}.$$

Since  $W$  satisfies T1, it follows as in Step 2 that  $W$  will satisfy T2 if  $\mathcal{K} \subset W$ . The next lemma verifies this.

LEMMA 3.8.  $\mathcal{K} \subset L$ .

*Proof.* Let  $w$  belong to  $\mathcal{K}$ . Then there are integers  $p \leq q$  such that  $\alpha^p(w) \in W_+$  and  $\alpha^q(w) \in W_-$ . Since  $W = V'L$  and, by Lemma 3.7,  $(V'L)_+ = V'_+L$ , there are  $u \in V'_+$  and  $l \in L$  with  $\alpha^p(w) = ul$ . Now  $\alpha^{q-p}(u) = \alpha^q(w)\alpha^{q-p}(l^{-1}) \in V'_-L$ , whence  $\{\alpha^n(u)\}_{n \geq 0}$  has an accumulation point. By Lemma 3.2,  $u$  belongs to  $L$  and it follows that  $w$  is in  $L$ . ■

It remains to show that  $[\alpha(W) : \alpha(W) \cap W] \leq [\alpha(V) : \alpha(V) \cap V]$ , with equality only if  $V$  satisfies T2. For this we recall from Step 1 that, since  $W$  and  $V$  both satisfy T1,  $[\alpha(W) : \alpha(W) \cap W] = [\alpha(W_+) : W_+]$  and  $[\alpha(V) : \alpha(V) \cap V] = [\alpha(V_+) : V_+]$ . Hence it will suffice to show that

$$[\alpha(W_+) : W_+] \leq [\alpha(V_+) : V_+] \quad (7)$$

with equality only if  $V$  satisfies T2.

LEMMA 3.9.  $[\alpha(V_+) : V_+] = [\alpha(V'_+) : V'_+]$ .

*Proof.* By definition  $V'_+ \subset V_+$  and, by Lemma 3.4, there is  $p \geq 0$  such that  $\alpha^{-p}(V_+) \subset V'_+$ . Hence

$$\alpha^{-p}(V'_+) \subset \alpha^{-p}(V_+) \subset V'_+ \subset V_+$$

and

$$\begin{aligned} [V_+ : \alpha^{-p}(V'_+)] &= [V_+ : V'_+][V'_+ : \alpha^{-p}(V'_+)] \\ &= [V_+ : \alpha^{-p}(V_+)] [\alpha^{-p}(V_+) : \alpha^{-p}(V'_+)] \end{aligned}$$

Since  $\alpha$  is an automorphism  $[V_+ : V'_+] = [\alpha^{-p}(V_+) : \alpha^{-p}(V'_+)]$  and so we have that  $[V'_+ : \alpha^{-p}(V'_+)] = [V_+ : \alpha^{-p}(V_+)]$ . The result follows because  $[V_+ : \alpha^{-p}(V_+)] = [\alpha(V_+) : V_+]^p$  and  $[V'_+ : \alpha^{-p}(V'_+)] = [\alpha(V'_+) : V'_+]^p$ . ■

LEMMA 3.10.  $[\alpha(V'_+L) : V'_+L] \leq [\alpha(V'_+) : V'_+]$  with equality if and only if  $L \subset V'_+$ .

*Proof.* Define a map  $\varphi: \alpha(V'_+)/V'_+ \rightarrow \alpha(V'_+L)/V'_+L$  by  $\varphi(uV'_+) = uV'_+L$ . Since  $\alpha(V'_+L) = \alpha(V'_+)L$ ,  $\varphi$  is well defined and surjective. Hence

$$|\alpha(V'_+L)/V'_+L| \leq |\alpha(V'_+)/V'_+|,$$

which is just the inequality to be proved.

It is clear that  $\varphi$  is injective if  $L \subset V'_+$ . Suppose that  $L \not\subset V'_+$ . Then  $\mathcal{L} \not\subset V'_+$  and, by acting with some power of  $\alpha$ , we may find  $w \in \mathcal{L} \cap \alpha(V'_+) \setminus V'_+$ . Then  $wV'_+ \neq V'_+$  and  $\varphi(wV'_+) = wV'_+L = V'_+L$  and so  $\varphi$  is not injective, thus completing the proof of the lemma. ■



Recalling that  $W_+ = V'_+ L$  and noting that  $L \subset V$  if and only if  $L \subset V'_+$ , Lemmas 3.9 and 3.10 imply (7) and the proof of Theorem 3.1 is complete.

The next result follows immediately from the theorem because a subgroup  $U$  is tidy for  $\alpha$  if and only if it is tidy for  $\alpha^{-1}$ .

**COROLLARY 3.11.** *Let  $\alpha$  be a continuous automorphism of the totally disconnected locally compact group  $G$  and let  $U$  be a compact open subgroup of  $G$ . Then  $[\alpha(U) : \alpha(U) \cap U]$  equals the minimum value  $s_G(\alpha)$  if and only if  $[\alpha^{-1}(U) : \alpha^{-1}(U) \cap U]$  equals the minimum value  $s_G(\alpha^{-1})$ . ■*

Is there a direct proof of the corollary which does not use tidy subgroups?

**Remark 3.12.** (a) The proof of Theorem 3.1 describes an algorithm for finding a subgroup tidy for  $\alpha$  and computing  $s_G(\alpha)$ . The algorithm is effective provided that effect can be given to the existence assertions in the applications of compactness made in the proof. If  $G$  is discrete, then the algorithm is complete after Step 1 because a finite subgroup  $V$  has been found satisfying  $\alpha(V) = V$ . Of course we know already in this case that  $s_G(\alpha) = 1$  and  $\{e\}$  is tidy. All the steps in this algorithm appear to be necessary, as can be seen by applying it to the subgroup  $U$  described in Example 6.1.

(b) This construction of tidy subgroups differs from the construction given in [6] in the definition of the subgroup  $V'$ . In [6] the subgroup playing the role of  $V'$  is defined to be  $\bigcap_{l \in L} lVl^{-1}$ . This subgroup is obviously normalised by  $L$  and so the equivalent of Lemma 3.3 is not required. However, the present construction is clearer and is also the “right” algorithm for finding tidy subgroups because at each step the index  $[\alpha(U) : \alpha(U) \cap U]$  is reduced. That is not the case for the construction in [6] because it can happen that  $[\alpha(V''_+) : V''_+] > [\alpha(V_+) : V_+]$ , where  $V''$  denotes  $\bigcap_{l \in L} lVl^{-1}$ ; see Example 6.3.

#### 4. THE SCALE FUNCTION ON SUBGROUPS AND QUOTIENT GROUPS

Tidy subgroups and the scale function are natural features of the structure of a totally disconnected group but as yet their behaviour under group homomorphisms is not understood. Herman has investigated the properties of the scale function on subgroups and quotient groups in his thesis [1]. Results in this section extend some of his work.

**LEMMA 4.1.** *Let  $G$  be a totally disconnected locally compact group, let  $\alpha$  be a continuous automorphism of  $G$ , and let  $U$  be a compact open subgroup*

of  $G$  which is tidy for  $\alpha$ . Suppose that  $H$  is a closed subgroup of  $G$  such that  $\alpha(H) = H$  and put  $U' = U \cap H$ . Then  $U'$  is a compact open subgroup of  $H$  and there is an integer  $n$  such that  $\bigcap_{k=0}^n \alpha^k(U')$  is tidy for  $\alpha|_H$ .

*Proof.* It is clear that  $U'$  is a compact open subgroup of  $H$ . [6, Lemma 1] shows that there is an integer  $n$  such that  $\bigcap_{k=0}^n \alpha^k(U') \stackrel{\text{def}}{=} V$  satisfies T1. Now  $V_+ = U'_+$ , whence  $V_{++} = U'_{++} = U_{++} \cap H$ , which is closed. Hence  $V$  satisfies T2 and is tidy for  $\alpha|_H$ . ■

Since  $\bigcap_{k=0}^n \alpha^k(U)$  is tidy for  $\alpha$  and  $H \cap \bigcap_{k=0}^n \alpha^k(U)$  equals  $V$ , tidiness is sometimes preserved when restricting to subgroups.

**COROLLARY 4.2.** *Let  $G$  be a totally disconnected locally compact group, let  $\alpha$  be a continuous automorphism, and let  $H$  be a closed subgroup of  $G$  such that  $\alpha(H) = H$ . Then there is a compact open subgroup  $U$  of  $G$  which is tidy for  $\alpha$  and such that  $H \cap U$  is tidy for  $\alpha|_H$ .*

Example 6.4 shows, however, that it is not always the case that  $H \cap U$  is tidy for  $\alpha|_H$  when  $U$  is tidy for  $\alpha$ . Even so, the corollary does allow the following comparison between the scale function on  $G$  and its restriction to a subgroup. This extends [1, Theorems 4.2 and 4.5].

**PROPOSITION 4.3.** *Let  $H$  be a closed subgroup of the totally disconnected locally compact group  $G$  and let  $\alpha$  be a continuous automorphism of  $G$  such that  $\alpha(H) = H$ . Then*

$$s_H(\alpha|_H) \leq s_G(\alpha).$$

*Proof.* Choose  $U$  such that  $U$  and  $H \cap U$  are tidy for  $\alpha$  and  $\alpha|_H$  respectively. Define  $\varphi: \alpha(H \cap U_+)/ (H \cap U_+) \rightarrow \alpha(U_+)/U_+$  by  $\varphi(u(H \cap U_+)) = uU_+$ , ( $u \in \alpha(H \cap U_+)$ ). Since  $s_H(\alpha|_H) = |\alpha(H \cap U_+)/ (H \cap U_+)|$  and  $s_G(\alpha) = |\alpha(U_+)/U_+|$ , it will suffice to show that  $\varphi$  is injective.

Suppose that  $\varphi(u(H \cap U_+)) = \varphi(v(H \cap U_+))$  for some  $u, v$  in  $\alpha(H \cap U_+)$ . This means that there is  $x \in U_+$  with  $u = vx$ , but then  $x = v^{-1}u$  and so  $x$  belongs to  $H$  as well. Hence  $u(H \cap U_+) = v(H \cap U_+)$ . ■

The above proof shows that  $\varphi$  is a bijection between  $\alpha(H \cap U_+)/ (H \cap U_+)$  and  $\alpha(H \cap U_+)U_+/U_+$  and so  $s_H(\alpha|_H)$  can be computed using subgroups of  $G$ .

**COROLLARY 4.4.** *Under the hypotheses of the proposition we have*

$$s_H(\alpha|_H) = [\alpha(H \cap U_+)U_+ : U_+].$$

It might be thought that  $s_H(\alpha|_H)$  should divide  $s_G(\alpha)$  when  $H$  is a closed subgroup of  $G$ , but Example 6.2 shows that that is not the case in general. However, it is so when  $H$  is a normal subgroup of  $G$ . Some further notation is required for this.

Let  $H$  be a closed normal subgroup of  $G$ . Then  $q: x \mapsto xH: G \rightarrow G/H$  will denote the quotient map. A continuous automorphism  $\alpha$  of  $G$  which leaves  $H$  invariant induces an automorphism  $\dot{\alpha}$  of  $G/H$  by  $\dot{\alpha}(xH) = \alpha(x)H$ .

LEMMA 4.5. *Let  $G$  be a totally disconnected locally compact group  $G$ , let  $H$  be a closed normal subgroup of  $G$ , and let  $\alpha$  be a continuous automorphism of  $G$  such that  $\alpha(H) = H$ . Then there is a subgroup,  $U$ , of  $G$  which is tidy for  $\alpha$  and such that*

(1)  $U \cap H$  is tidy for  $\alpha|_H$  and

(2)  $q(U)$  satisfies T1 for  $\dot{\alpha}$  and  $Lq(U) = q(U)L$ , where  $L$  is the closure of

$$\{wH \in G/H : \dot{\alpha}^n(wH) \in q(U) \text{ for all but finitely many } n\}. \quad (8)$$

*Proof.* Choose  $V < G$  tidy for  $\alpha$  and such that  $V \cap H$  is tidy for  $\alpha|_H$ . Then

$$\begin{aligned} q(V_{\pm}) &= q\left(\bigcap_{n \geq 0} \alpha^{\pm n}(V)\right) \\ &\subset \bigcap_{n \geq 0} q(\alpha^{\pm n}(V)) = \bigcap_{n \geq 0} \dot{\alpha}^{\pm n}(q(V)) = q(V)_{\pm}. \end{aligned}$$

Since  $V$  satisfies T1, we have  $q(V) \subset q(V_+)q(V_-) \subset q(V)_+q(V)_-$ , so that  $q(V)$  satisfies T1.

As in Eq. (4) in the proof of Theorem 3.1, define  $L$  to be the closure of

$$\{wH \in G/H : \dot{\alpha}^n(wH) \in q(V) \text{ for all but finitely many } n\}.$$

and

$$q(V)' = \{wH \in q(V) : (lH)(wH)(lH)^{-1} \in q(V)L \text{ for all } lH \in L\}.$$

Now set

$$U = V \cap q^{-1}(q(V)').$$

Then  $q(U) = q(V)'$  and so  $q(U)$  satisfies T1 by Lemma 3.6 and  $q(U)L = Lq(U)$  by Lemma 3.3. Also note that Lemma 3.4 implies that  $L$  satisfies (8). Since  $H$  is the kernel of  $q$ , we have  $U \cap H = V \cap H$ , which is tidy for  $\alpha|_H$ . It remains to show that  $U$  is tidy for  $\alpha$ .

Let  $u$  belong to  $U$ . Then  $u = vw$ , where  $v \in V_+$  and  $w \in V_-$ . Hence  $q(u)$  belongs to  $q(V)'$  and  $q(u) = q(v)q(w)$  where  $q(v) \in q(V)_+$  and  $q(w) \in q(V)_-$ . The proof of Lemma 3.6 shows that in fact  $q(v)$  and  $q(w)$  belong to  $q(V)_+$  and  $q(V)_-$  respectively. Then  $q(\alpha^{-n}(v)) = \dot{\alpha}^{-n}(q(v))$  belongs to  $q(V)'$  for each  $n \geq 0$ , whence  $\alpha^{-n}(v)$  is in  $U$  for each  $n \geq 0$  and  $v \in U_+$ . That  $w \in U_-$  follows similarly and so  $U$  satisfies T1.

Finally, since  $V$  satisfies T2 and  $U \subset V$ ,

$$\{u \in U : \alpha^n(u) \in U \text{ for all but finitely many } n\} \subset V_+ \cap V_-.$$

Clearly,  $q(V_+ \cap V_-) \subset L$ . Hence  $q(V_+ \cap V_-) \subset q(V)'$  and, consequently,  $V_+ \cap V_- \subset U$ . Therefore  $U$  satisfies T2. ■

As seen in the proof of Theorem 3.1,  $q(U)L$  is tidy for  $\dot{\alpha}$ , but it is not possible in general to find a compact open subgroup  $U$  of  $G$  such that  $U$  is tidy for  $\alpha$  and  $q(U)$  is tidy for  $\dot{\alpha}$ ; see Example 6.5.

LEMMA 4.6. *Let  $G, H, \alpha$ , and  $U$  be as in the previous lemma. Then there is a closed subgroup  $J < G$  such that  $\alpha(H \cap U_+)U_+ < J < \alpha(U_+)$  and  $s_{G/H}(\dot{\alpha}) = [\alpha(U_+) : J]$ .*

*Proof.* The group  $q(U)L < G/H$ , where  $L$  is as defined in the previous lemma, is tidy for  $\dot{\alpha}$ . Hence  $s_{G/H}(\dot{\alpha}) = [\dot{\alpha}(q(U)_+)L : q(U)_+L]$ .

Although  $q(U)_+L$  need not be normal in  $\dot{\alpha}(q(U)_+)L$ , the proof of the first isomorphism theorem for groups shows that there is a bijection

$$\dot{\alpha}(q(U)_+)/(\dot{\alpha}(q(U)_+) \cap q(U)_+L) \rightarrow \dot{\alpha}(q(U)_+L)/q(U)_+L.$$

Hence  $s_{G/H}(\dot{\alpha}) = [\dot{\alpha}(q(U)_+) : \dot{\alpha}(q(U)_+) \cap q(U)_+L]$ . Now  $q$  restricts to a homomorphism  $\tilde{q} : \alpha(U_+) \rightarrow \dot{\alpha}(q(U)_+)$ . Define

$$J = \tilde{q}^{-1}(\dot{\alpha}(q(U)_+) \cap q(U)_+L),$$

a closed subgroup of  $\alpha(U_+)$ . To see that  $\alpha(H \cap U_+)U_+ < J$ , note that, since  $\alpha(H \cap U_+) < H$ , we have  $q(\alpha(H \cap U_+)U_+) = q(U_+)$  and then note that  $q(U_+) < \dot{\alpha}(q(U)_+) \cap q(U)_+L$ .

It follows immediately from the definition of  $J$  that

$$[\alpha(U_+) : J] \leq [\dot{\alpha}(q(U)_+) : \dot{\alpha}(q(U)_+) \cap q(U)_+L].$$

For the proof of the inequality in the other direction, consider  $\dot{\alpha}(q(u)) \in \dot{\alpha}(q(U)_+)$ . Since  $U$  is tidy, there are  $v$  and  $w$  in  $U_+$  and  $U_-$  respectively such that  $u = vw$ . Then  $\dot{\alpha}(q(u)) = q(\alpha(v))q(\alpha(w))$  where  $q(\alpha(v)) \in q(\alpha(U_+))$ . It follows that  $q(\alpha(w)) \in \dot{\alpha}(q(U)_+)$  and then, since  $\alpha(w) \in U$ , that  $q(\alpha(w)) \in q(U)_+$ . Hence each coset in  $\dot{\alpha}(q(U)_+)/(\dot{\alpha}(q(U)_+) \cap q(U)_+L)$  has a representative belonging to  $q(\alpha(U_+))$  and so

$$[\alpha(U_+) : J] \geq [\dot{\alpha}(q(U)_+) : \dot{\alpha}(q(U)_+) \cap q(U)_+L].$$

■

[1, Theorem 4.8] shows that if the scale function on  $G$  is identically one then the same is true on any quotient of  $G$ . We can now extend this result as follows.

**PROPOSITION 4.7.** *Let  $G$  be a totally disconnected locally compact group and let  $H$  be a closed normal subgroup of  $G$ . Then  $s_H(\alpha|_H)s_{G/H}(\dot{\alpha})$  divides  $s_G(\alpha)$ .*

*Proof.* Choose a compact open subgroup  $U$  of  $G$  satisfying the conditions guaranteed by Lemma 4.5.

Since  $H$  is normal in  $G$ ,  $\alpha(H \cap U_+)$  is normal in  $\alpha(U_+)$  and  $\alpha(H \cap U_+)U_+$  is a group. Hence, by Lemma 4.6, we have the inclusion of subgroups

$$U_+ < \alpha(H \cap U_+)U_+ < J < \alpha(U_+)$$

and it follows that

$$\begin{aligned} s_G(\alpha) &= [\alpha(U_+) : U_+] \\ &= [\alpha(U_+) : J][J : \alpha(H \cap U_+)U_+][\alpha(H \cap U_+)U_+ : U_+]. \end{aligned}$$

By Corollary 4.3,  $[\alpha(H \cap U_+)U_+ : U_+] = s_H(\alpha|_H)$  and Lemma 4.6 shows that  $[\alpha(U_+) : J] = s_{G/H}(\dot{\alpha})$ . ■

The next result applies to the scale function on groups rather than automorphisms. Recall that the scale function on  $G$  is given by  $s_G(x) = s_G(\alpha_x)$  ( $x \in G$ ), where  $\alpha_x$  is the inner automorphism  $\alpha_x: y \mapsto xyx^{-1}$ .

**PROPOSITION 4.8.** *Let  $H$  be a closed normal subgroup of  $G$ . Then  $s_H = s_{G/H}$ .*

*Proof.* Let  $x$  be in  $H$ . Choose a subgroup  $U < G$  which is tidy for  $\alpha_x$  and such that  $H \cap U$  is tidy for  $\alpha_x|_H$ . Then for each  $u \in U_+$  we have

$$(xux^{-1})u^{-1} = x(ux^{-1}u^{-1}),$$

where the left side belongs to  $xU_+x^{-1}$  and the right side to  $H$  because  $H$  is normal. Hence  $(xux^{-1})u^{-1}$  belongs to  $H \cap xU_+x^{-1} = x(H \cap U_+)x^{-1}$  and it follows that  $xux^{-1} \in x(H \cap U_+)x^{-1}U_+$ . Therefore  $[xU_+x^{-1} : U_+] = [x(H \cap U_+)x^{-1}U_+ : U_+]$  and Corollary 4.4 shows that  $s_H(x) = s_G(x)$ . ■

The modular function on  $G$  satisfies the identity

$$\Delta_G(x) = \Delta_H(\alpha_x)\Delta_{G/H}(xH) \quad (x \in G),$$

whenever  $H$  is a closed normal subgroup of  $G$ ; see [2, (15.23)]. Propositions 4.7 and 4.8 are the analogues of this for the scale function. It is not true that  $s_G(\alpha) = s_H(\alpha|_H)s_{G/H}(\dot{\alpha})$  in general; see Example 6.4.

## 5. INCREASING UNIONS AND INVERSE LIMITS OF GROUPS

The behaviour of the scale of an automorphism  $\alpha$  of  $G$  on an increasing union  $\bigcup_{n \geq 0} H_n$  of closed subgroups of  $G$  such that, for each  $n$ ,  $\alpha(H_n) = H_n$  is described below. An example of such an automorphism is the inner automorphism  $\alpha_x$  where  $x$  belongs to  $H_1$ . For such automorphisms we may speak of the uniform behaviour of  $s_{H_n}(\alpha_x|_{H_n})$  for  $x$  in compact subsets of  $H_1$  and this is done in Proposition 5.3.

The results shown here for increasing unions could be proved for directed unions of closed subgroups of  $G$ , i.e., for unions of families  $\{H_i\}_{i \in I}$  such that for each pair  $i_1, i_2 \in I$  there is  $i_3 \in I$  such that  $H_{i_1} \cup H_{i_2} \subset H_{i_3}$ , but they would be slightly more complicated to state.

**LEMMA 5.1.** *Let  $H$  be a closed subgroup of the totally disconnected locally compact group  $G$ . Let  $\alpha \in \text{Aut}(G)$  satisfy  $\alpha(H) = H$ ,  $s_H(\alpha|_H) = s_G(\alpha)$ , and  $s_H(\alpha^{-1}|_H) = s_G(\alpha^{-1})$ . Let  $U$  be a compact open subgroup of  $G$  which is tidy for  $\alpha$  and with  $V = U \cap H$  tidy for  $\alpha|_H$ . Then  $U = V(U_+ \cap U_-)$ .*

*Proof.* Since  $U$  satisfies T1, [6, proof of Lemma 1] shows that for each  $n \geq 0$  we have  $\alpha^n(U) = \alpha^n(U_+) (\bigcap_{k=0}^n \alpha^k(U))$ . Since  $s_H(\alpha|_H) = s_G(\alpha)$  and  $U$  and  $V$  are tidy, the map  $\psi: \alpha^n(V_+)/V_+ \rightarrow \alpha^n(U)/(\bigcap_{k=0}^n \alpha^k(U))$  defined by  $\psi(vV_+) = v(\bigcap_{k=0}^n \alpha^k(U))$  is a bijection. Hence for each  $u \in U$  the compact set

$$C_n(u) \stackrel{\text{def}}{=} \left\{ v \in V_+ : \alpha^n(v^{-1}u) \in \bigcap_{k=0}^n \alpha^k(U) \right\}$$

is not empty. As in [6, proof of Lemma 1], we have  $C_{n+1}(u) \subset C_n(u)$  for each  $n$ , so that  $\bigcap_{n \geq 0} C_n(u) \neq \emptyset$ .

Given  $u \in U$ , choose  $v \in \bigcap_{n \geq 0} C_n(u)$ . Then  $u = vu'$ , where  $v \in V_+$  and  $u' \in \bigcap_{k \geq 0} \alpha^{-k}(U) = U_-$ . Repeating this argument for  $u'$  and  $\alpha^{-1}$  we find  $w \in V_-$  and  $u'' \in U_+ \cap U_-$  such that  $u' = wu''$ . Then  $u = (vw)u''$  where  $vw \in V$  and  $u'' \in U_+ \cap U_-$ . ■

**LEMMA 5.2.** *Let  $\{H_n\}$  be an increasing sequence of closed subgroups of the totally disconnected locally compact group  $G$ . Let  $\alpha \in \text{Aut}(G)$  be such that  $\alpha(H_n) = H_n$  for each  $n$  and  $s_{H_n}(\alpha|_{H_n})$  and  $s_{H_n}(\alpha^{-1}|_{H_n})$  are independent of  $n$ . Then there are subgroups  $V_n < H_n$  tidy for  $\alpha|_{H_n}$  and such that  $V_n \cap H_1 = V_1$  for each  $n$ .*

*Proof.* Choose  $U < G$  which is tidy for  $\alpha$ . By Lemma 4.1 we may suppose that  $U \cap H_1$  is tidy for  $\alpha|_{H_1}$ . Set  $V_1 = U \cap H_1$ .

Let  $n$  be greater than 1. By Lemma 4.1 there is  $r \geq 0$  such that  $\cap_{k=0}^r \alpha^k(U) \cap H_n$  is tidy for  $\alpha|_{H_n}$ . By Lemma 5.1,

$$\bigcap_{k=0}^r \alpha^k(U) \cap H_n = \left( \bigcap_{k=0}^r \alpha^k(U) \cap H_1 \right) L_n = \left( \bigcap_{k=0}^r \alpha^k(V_1) \right) L_n,$$

where  $L_n \stackrel{\text{def}}{=} (\cap_{k=0}^r \alpha^k(U) \cap H_n)_+ \cap (\cap_{k=0}^r \alpha^k(U) \cap H_n)_-$ . Hence

$$\left( \bigcap_{k=0}^r \alpha^k(V_1) \right) L_n = L_n \left( \bigcap_{k=0}^r \alpha^k(V_1) \right).$$

Now let  $u \in (\cap_{k=0}^r \alpha^k(V_1))_+$  and  $l \in L_n$ . Then  $ulu^{-1} = vwm$ , where  $v \in (\cap_{k=0}^r \alpha^k(V_1))_+$ ,  $w \in (\cap_{k=0}^r \alpha^k(V_1))_-$  and  $m \in L_n$ . Since  $w = v^{-1}ulu^{-1}m^{-1}$ ,  $\{\alpha^{-n}(w)\}_{n \geq 0}$  has an accumulation point and so, by Lemma 3.2,

$$w \in \left( \bigcap_{k=0}^r \alpha^k(V_1) \right)_+ \cap \left( \bigcap_{k=0}^r \alpha^k(V_1) \right)_- \subset L_n.$$

It follows that

$$\left( \bigcap_{k=0}^r \alpha^k(V_1) \right)_+ L_n = L_n \left( \bigcap_{k=0}^r \alpha^k(V_1) \right)_+.$$

By noting that  $(\cap_{k=0}^r \alpha^k(V_1))_+ = (V_1)_+$ , this may be written more briefly as

$$(V_1)_+ L_n = L_n (V_1)_+. \quad (9)$$

Now  $(\cap_{k=0}^r \alpha^{-k}(U)) \cap H_n = \alpha^{-r}(\cap_{k=0}^r \alpha^k(U) \cap H_n)$  is tidy for  $\alpha|_{H_n}$  and

$$\left( \bigcap_{k=0}^r \alpha^{-k}(U) \cap H_n \right)_+ \cap \left( \bigcap_{k=0}^r \alpha^{-k}(U) \cap H_n \right)_- = \alpha^{-r}(L_n) = L_n.$$

A similar argument to that above, using this and noting that  $(\cap_{k=0}^r \alpha^{-k}(V_1))_- = (V_1)_-$ , shows that

$$(V_1)_- L_n = L_n (V_1)_-. \quad (10)$$

Define

$$V_n = V_1 L_n = (V_1)_+ (V_1)_- L_n.$$

Then Eqs. (9) and (10) and the fact that  $(V_1)_+ (V_1)_- = (V_1)_- (V_1)_+$  imply that  $V_n$  is a subgroup of  $H_n$  and it is compact because  $V_1$  and  $L_n$  are.

Since  $V_n \supset \bigcap_{k=0}^r \alpha^k(U) \cap H_n$ ,  $V_n$  is an open subgroup of  $H_n$ . It is easily verified that  $(V_n)_\pm = (V_1)_\pm L_n$  and thence that  $V_n$  is tidy for  $\alpha|_{H_n}$ . That  $V_n \cap H_1 = V_1$  is also easily checked. ■

It is a necessary step in the proof that  $V_1$  should be chosen by intersecting  $H_1$  with a tidy subgroup of  $G$  as the conclusion may not hold for arbitrary tidy subgroups of  $H_1$ . Let  $H$  be a closed subgroup of  $G$ , let  $\alpha \in \text{Aut}(G)$  satisfy  $\alpha(H) = H$ , and let the values of the respective scale functions agree. Then it is not in general the case that every tidy subgroup of  $H$  is the intersection of  $H$  with a tidy subgroup of  $G$ ; see Example 6.7.

We now specialise to the case where the automorphism is conjugation by  $x$  for some  $x$  in  $H_1$  and regard the scale as a function on  $G$ . Thus  $s(\alpha_x)$  will be denoted by  $s(x)$ .

**PROPOSITION 5.3.** *Let  $\{H_n\}$  be an increasing sequence of closed subgroups of the totally disconnected locally compact group  $G$ . Then for each  $x \in H_1$  the limit  $\lim_{n \rightarrow \infty} s_{H_n}(x)$  exists and the convergence is uniform on compact subsets of  $H_1$ .*

*Proof.* Proposition 4.3 implies that for each  $x$  in  $H_1$   $\{s_{H_n}(x)\}_{n \geq 0}$  is an increasing sequence of integers bounded by  $s_G(x)$ . Hence  $\lim_{n \rightarrow \infty} s_{H_n}(x)$  exists. In fact,  $\{s_{H_n}(x)\}_{n \geq 0}$  is constant after finitely many terms.

Let  $K$  be a compact subset of  $H_1$ . Then for each  $x \in K$  there is an integer  $N(x)$  such that  $s_{H_n}(x)$  and  $s_{H_n}(x^{-1})$  are constant for  $n \geq N(x)$ . Lemma 5.2 shows that there are, for  $n \geq N(x)$ , compact open sets  $V_n^{(x)} < H_n$  tidy for  $x$  and such that  $V_n^{(x)} \cap H_{N(x)} = V_{N(x)}^{(x)}$ . Put  $V_1^{(x)} = V_{N(x)}^{(x)} \cap H_1$ —a compact open subgroup of  $H_1$ .

[6, Theorem 3] shows that  $s_{H_n}$  is constant on  $xV_n^{(x)} \subset H_n$ . In particular,  $s_{H_n}$  is constant on the open neighbourhood  $xV_1^{(x)}$  of  $x$  in  $H_1$ . As shown in the last paragraph, the value of  $s_{H_n}$  on  $xV_1^{(x)}$  is also independent of  $n$  for  $n \geq N(x)$ . Now  $\{xV_1^{(x)} : x \in K\}$  is an open cover of  $K$  which, by the compactness of  $K$ , has a finite subcover  $\{x_i V_1^{(x_i)} : i = 1, 2, \dots, m\}$ . Put  $N = \max\{N(x_i) : i = 1, 2, \dots, m\}$ . Then  $s_{H_n}(x)$  is constant for  $n \geq N$  and  $x \in K$ . Thus the convergence is uniform on  $K$ . ■

Let  $\{H_n\}$  be an increasing sequence of closed subgroups of  $G$  and suppose that  $\bigcup_{n \geq 0} H_n$  is dense in  $G$ . Then  $\lim_{n \rightarrow \infty} s_{H_n}(x)$  exists for each  $x \in H_1$ . However, it is not necessarily the case that this limit is equal to  $s_G(x)$ ; see Example 6.6.

The next result is included following the suggestion of Helge Glöckner.

**PROPOSITION 5.4.** *Let the totally disconnected locally compact group  $G = \lim_{\leftarrow} G_i$  be an inverse limit of groups  $G_i$  where each of the maps*



$q_{ij}: G_i \rightarrow G_j$  and the limit homomorphisms  $q_i: G \rightarrow G_i$  is onto. Then for every  $x \in G$

$$\lim_i s_{G_i}(q_i(x)) = s_G(x),$$

where the convergence is uniform on compact subsets of  $G$ .

*Proof.* Let  $x$  be in  $G$  and choose  $U < G$  tidy for  $x$ . Then since  $G$  is the inverse limit of  $\{G_i\}$  and since  $U$  is open, there is an  $i$  such that  $\ker(q_i) < U$ . Since the kernel of a homomorphism is normal,  $\ker(q_i) < U_+ \cap U_-$  and it follows that  $q_i(U)$  is tidy for  $q_i(x)$  and that  $s_{G_i}(q_i(x)) = s_G(x)$ . The same holds for every  $j \geq i$  because  $\ker(q_j) < \ker(q_i)$ ; hence  $\lim_i s_{G_i}(q_i(x)) = s_G(x)$ .

That convergence is uniform on compact subsets of  $G$  follows as in Proposition 5.3. ■

## 6. EXAMPLES

The examples in this section show that many of the above results are sharp and that certain tempting simplifications of proofs cannot work. Each of the examples is a restricted product over the integers of copies of a starter group. Usually this starter group is finite but it is infinite in one case. The identity element of the starter group is denoted by  $\iota$  while the identity in the larger product group is denoted by  $e$ .

EXAMPLE 6.1. Let  $S_3 = \{\iota, \tau, \tau^2, \sigma_1, \sigma_2, \sigma_3\}$  be the symmetric group on three letters. Define  $G$  to be the subgroup of the infinite product  $S_3^{\mathbb{Z}}$

$$G = \{f \in S_3^{\mathbb{Z}} : \exists N \text{ such that } f(n) \in \{\iota, \sigma_1\} \quad \forall n \leq N\}.$$

For each  $N$  define the subgroup

$$U_N = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq -N, f(n) = \iota \text{ if } -N < n \leq N\}.$$

Define a totally disconnected locally compact topology on  $G$  by taking as a neighbourhood base  $\{fU_N : f \in G, N \geq 0\}$ . Then the subgroup  $U_N$  is compact and open and is isomorphic to the product  $(\prod_{n \leq -N} \{\iota, \sigma_1\}) \times (\prod_{n \geq N} S_3)$  with the product topology.

Define a continuous automorphism  $\alpha$  of  $G$  by

$$\alpha(f)(n) = f(n+1) \quad (f \in G, n \in \mathbb{Z}).$$

The method for finding tidy subgroups described in Theorem 3.1 will now be illustrated by using it to find a subgroup tidy for  $\alpha$ .

We start by choosing the compact open subgroup

$$U = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq 1, f(2) = \iota, \text{ and } f(5) \in \{\iota, \sigma_2\}\}.$$

Then

$$U \cap \alpha(U) = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq 0, f(1), f(2) = \iota, \\ \text{and } f(4), f(5) \in \{\iota, \sigma_2\}\}$$

so that  $[\alpha(U) : U \cap \alpha(U)] = |S_3| [S_3 : \{\iota, \sigma_2\}] = 18$ . Also,

$$U_+ = \{f \in G : f(n) = \iota \text{ if } n \leq 2 \text{ and } f(n) \in \{\iota, \sigma_2\} \text{ if } 3 \leq n \leq 5\}$$

and

$$U_- = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq 1 \text{ and } f(n) = \iota \text{ if } n \geq 2\}$$

so that  $U_+ U_- \neq U$ .

*Step 1.* Let  $V = U \cap \alpha(U) \cap \alpha^2(U)$ . Then

$$V = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq -1, \\ f(0), f(1), f(2) = \iota \text{ and } f(3), f(4), f(5) \in \{\iota, \sigma_2\}\},$$

$$V_+ = \{f \in G : f(n) = \iota \text{ if } n \leq 2 \text{ and } f(n) \in \{\iota, \sigma_2\} \text{ if } 3 \leq n \leq 5\},$$

and

$$V_- = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq -1 \text{ and } f(n) = \iota \text{ if } n \geq 0\}$$

so that  $V_+ V_- = V$ . Also the index

$$[\alpha(V) : V \cap \alpha(V)] = [\alpha(V_+) : V_+] = |\{\iota, \sigma_2\}| [S_3 : \{\iota, \sigma_2\}] = 6$$

has been reduced.

*Step 2.* It is easily checked that

$$\mathcal{L} = \{f \in G : f(n) = \sigma_1 \text{ for finitely many } n \text{ and } f(n) = \iota \text{ otherwise}\}$$

and  $L = \{\iota, \sigma_1\}^{\mathbb{Z}}$ . Since  $L \not\subset V$ ,  $V$  does not satisfy T2. This may be seen directly because

$$V_{++} = \{f \in G : \exists N \text{ such that } f(n) = \iota \text{ for all } n \leq N\},$$

which is not closed.

*Step 3.* The set  $\{\iota, \sigma_1\}\{\iota, \sigma_2\}$  is not a subgroup of  $S_3$  and so  $VL$  is not a group. Since  $\sigma_1\sigma_2\sigma_1^{-1} = \sigma_3$  does not belong to this set, the group  $V'$  equals

$$\{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq -1 \text{ and } f(n) = \iota \text{ if } 0 \leq n \leq 5\}.$$

Now

$$W = V'L = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq 5\}$$

satisfies  $W_+ = W$  and  $W_- = L$ , so that  $W = W_+W_-$  and  $W_{++} = G$  and  $W_{--} = L$  are closed. Hence  $W$  is tidy and  $s(\alpha) = [\alpha(W) : W \cap \alpha(W)] = 3$ . The index is reduced again when passing from  $V$  to  $W$ .

In this example the argument given in [6] produces the same groups at corresponding steps as the argument used in Theorem 3.1 because  $\bigcap_{l \in L} lVl^{-1}$  equals the  $V'$  defined here. Example 6.3 is a case where the two arguments produce different groups at corresponding steps.

**EXAMPLE 6.2.** Let  $G$ ,  $\alpha$ , and  $W$  be as in the previous example and let  $H$  be the closed subgroup

$$H = \{f \in G : f(n) \in \{\iota, \sigma_2\} \text{ for all } n\}.$$

Then it is easily checked that

$$W' = H \cap W = \{f \in H : f(n) = \iota \text{ for all } n \leq 5\}$$

is a tidy subgroup for  $\alpha|_H$  and that  $s_H(\alpha|_H) = 2$ . Hence  $s_H(\alpha|_H)$  does not divide  $s_G(\alpha)$  in this case.

**EXAMPLE 6.3.** Let  $G$  be the compact group  $G = S_3^{\mathbb{Z}}$  with the product topology and let  $V$  be the subgroup

$$V = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } 1 \leq n \leq 3\}.$$

Let  $\alpha$  be the automorphism  $\alpha(f)(n) = f(n+1)$ . Then  $V_+ = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq 3\}$ ,  $V_- = \{f \in G : f(n) \in \{\iota, \sigma_1\} \text{ if } n \geq 1\}$ , and  $V$  satisfies T1. However,

$$V_{++} = \{f \in G : \exists N \text{ such that } f(n) \in \{\iota, \sigma_1\} \text{ if } n \leq N\}$$

is not closed and so  $V$  does not satisfy T2.

Now  $L = G$ , whence  $V' = V$  and  $W = V'L = G$ , which is tidy. However, since  $\{\iota, \sigma_1\}$  is not normal in  $S_3$ ,

$$V'' = \bigcap_{l \in L} lVl^{-1} = \{f \in G : f(n) = \iota \text{ if } 1 \leq n \leq 3\}.$$

Hence in this case the argument from [6] produces a different subgroup to the argument used here. Note that  $[\alpha(V):V \cap \alpha(V)] = 3$  whereas  $[\alpha(V''):V'' \cap \alpha(V'')] = 6$ , so that the argument from [6] does not decrease the index at every stage.

EXAMPLE 6.4. Let  $F$  be the abelian group  $\{\iota, a, b, ab\}$  isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Define  $G$  to be the subgroup of the infinite product  $F^{\mathbb{Z}}$

$$G = \{f \in F^{\mathbb{Z}} : \exists N \text{ such that } f(n) \in \{\iota, a\} \text{ and}$$

$$f(-n) \in \{\iota, b\} \text{ whenever } n \geq N\}.$$

For each  $m$  in  $\mathbb{Z}$  define the subgroups

$$A_m = \{f \in G : f(k) \in \{\iota, a\} \text{ if } k \geq m, f(k) = \iota \text{ otherwise}\}$$

and

$$B_m = \{f \in G : f(k) \in \{\iota, b\} \text{ if } k \leq m, f(k) = \iota \text{ otherwise}\}.$$

Define a totally disconnected locally compact topology on  $G$  by taking as a neighbourhood base  $\{f(A_m \times B_n) : f \in G, m, n \in \mathbb{Z}\}$ . Then the subgroups  $A_m \times B_n$  are compact, open, and isomorphic to the product  $(\prod_{k \geq m} \mathbb{Z}_2) \times (\prod_{k \leq n} \mathbb{Z}_2)$  with the product topology. The subgroups  $A = \bigcup_{m \in \mathbb{Z}} A_m$  and  $B = \bigcup_{m \in \mathbb{Z}} B_m$  are closed in  $G$  and  $G = A \times B$ .

As in earlier examples, let  $\alpha$  be the translation automorphism

$$\alpha(f)(n) = f(n+1) \quad (f \in G, n \in \mathbb{Z}).$$

It is easily seen that each of the subgroups  $A_m \times B_n$  is tidy for  $\alpha$ . We have  $(A_m \times B_n)_+ = A_m$ ,  $(A_m \times B_n)_{++} = A$ ,  $(A_m \times B_n)_- = B_n$ ,  $(A_m \times B_n)_{--} = B$ , and  $s_G(\alpha) = 2$ .

Let

$$H = \{f \in G : f(n) \in \{\iota, ab\} \text{ for every } n\}.$$

Then  $H$  is a closed subgroup of  $G$  and, since for each  $f \in H$  we have  $f(n) = \iota$  for all but finitely many  $n$ ,  $H$  is a countable discrete subgroup of  $G$ . In fact,  $H$  is isomorphic to the direct sum  $\sum_{\mathbb{Z}} \mathbb{Z}_2$ . The quotient  $G/H$  may be seen to be isomorphic to the compact group  $\mathbb{Z}_2^{\mathbb{Z}}$  with the product topology. For this, let  $\hat{q}: F \rightarrow \mathbb{Z}_2$  be the homomorphism given by  $\hat{q}(\iota) = \bar{0} = \hat{q}(ab)$  and  $\hat{q}(a) = \bar{1} = \hat{q}(b)$  and define  $q: G \rightarrow \mathbb{Z}_2^{\mathbb{Z}}$  by

$$q(f)(n) = \hat{q}(f(n)) \quad (f \in G, n \in \mathbb{Z}).$$

Then the kernel of  $q$  is  $H$  and  $q$  maps  $G$  onto  $\mathbb{Z}_2^{\mathbb{Z}}$ . Clearly,  $H$  is invariant under  $\alpha$  and  $\alpha|_H$  and  $\hat{\alpha}$  are just the translation automorphisms on  $\sum_{\mathbb{Z}} \mathbb{Z}_2$  and  $\mathbb{Z}_2^{\mathbb{Z}}$  respectively.

This example illustrates a number of points concerning tidy subgroups for normal subgroups and quotient groups.

1. Since  $H$  is discrete,  $s_H(\alpha|_H) = 1$  and, since  $G/H$  is compact,  $s_{G/H}(\dot{\alpha}) = 1$ . Hence  $s_G(\alpha) \neq s_H(\alpha|_H)s_{G/H}(\dot{\alpha})$ .

2. The subgroup  $A_0 \times B_0$  is tidy for  $\alpha$  but

$$H \cap (A_0 \times B_0) = \{f \in G : f(0) \in \{\iota, ab\} \text{ and } f(n) = \iota \text{ otherwise}\},$$

which is not tidy for  $\alpha|_H$  because it does not satisfy T1.

3. The subgroup  $A_0 \times B_2$  is tidy for  $\alpha$  but

$$q(A_0 \times B_2) = \{f \in \mathbb{Z}_2^{\mathbb{Z}} : f(1) = \iota\}$$

which is not tidy for  $\dot{\alpha}$  because it does not satisfy T2.

4. The subgroup  $A_0 \times B_1$  is tidy for  $\alpha$  and  $H \cap (A_0 \times B_1)$  and  $q(A_0 \times B_1)$  are tidy for  $\alpha|_H$  and  $\dot{\alpha}$  respectively.

EXAMPLE 6.5. This example uses the definitions and notation of the previous one. Let  $E = \langle a, b : a^2 = \iota = b^2 \rangle$ . Then  $E$  is isomorphic to the free product  $\mathbb{Z}_2 * \mathbb{Z}_2$ . Define  $G_1$  by

$$G_1 = \{f \in E^{\mathbb{Z}} : \exists N \text{ such that } f(n) \in \{\iota, a\} \text{ and } f(-n) \in \{\iota, b\} \text{ whenever } n \geq N\}.$$

For each  $m$  in  $\mathbb{Z}$  define the subgroups

$$A_{1,m} = \{f \in G_1 : f(k) \in \{\iota, a\} \text{ if } k \geq m, f(k) = \iota \text{ otherwise}\}$$

and

$$B_{1,m} = \{f \in G_1 : f(k) \in \{\iota, b\} \text{ if } k \leq m, f(k) = \iota \text{ otherwise}\},$$

and let  $\{f(A_{1,m} \times B_{1,n}) : f \in G, m, n \in \mathbb{Z}\}$  be a base of neighbourhoods for a totally disconnected locally compact topology on  $G_1$ . The sets  $A_{1,m} \times B_{1,n}$  are compact open subgroups of  $G_1$  for  $m > n$  but are not subgroups for  $m \leq n$ . Note that, if  $m \leq n$ , then the subgroup generated by  $A_{1,m} \times B_{1,n}$  is not compact because it contains a closed subgroup isomorphic to  $\mathbb{Z}_2 * \mathbb{Z}_2$ .

Let  $\alpha$  be the translation automorphism. Then each of the subgroups  $A_{1,m} \times B_{1,n}$  ( $m > n$ ) is tidy for  $\alpha$  and  $s_{G_1}(\alpha) = 2$ . It is important for this example to note that they are the only subgroups tidy for  $\alpha$ .

The subgroup  $H_1$  will be defined to be the kernel of the composition of certain homomorphisms. For the first homomorphism, let  $F$  and  $G$  be as

in the previous example and let  $\hat{r}: E \rightarrow F$  be the homomorphism which sends  $a \mapsto a$  and  $b \mapsto b$ . Define  $r: G_1 \rightarrow G$  by

$$r(f)(n) = \hat{r}(f(n)) \quad (f \in G_1, \quad n \in \mathbb{Z}).$$

The second homomorphism is defined as follows. Write  $G = A \times B$  as in the previous example and let  $t$  be the automorphism of  $G$  such that  $t|_A$  is the identity map and  $t|_B$  is the shift

$$t|_B(f)(n) = f(n+1) \quad (f \in B, \quad n \in \mathbb{Z}).$$

The third homomorphism is the map  $q: G \rightarrow \mathbb{Z}_2^{\mathbb{Z}}$  described in the previous example. Then  $H_1$  is the kernel of the homomorphism  $q \circ t \circ r: G_1 \rightarrow \mathbb{Z}_2^{\mathbb{Z}}$ . Each of  $r$ ,  $t$ , and  $q$  commutes with  $\alpha$  (on  $G$  or  $G_1$ ) and so  $\alpha(H_1) = H_1$ .

Now let  $U$  be a subgroup of  $G_1$  which is tidy for  $\alpha$ . Then  $U = A_{1,m} \times B_{1,n}$  where  $m > n$ . Hence  $r(U) = A_m \times B_n \subset G$  where  $m > n$ . Hence  $t \circ r(U) = A_m \times B_{n-1} \subset G$  where  $m > n$ . Hence  $q \circ t \circ r(U) = A_m \times B_{n-1} \subset \mathbb{Z}_2^{\mathbb{Z}}$  where  $m > n$ . Since  $m > n$ , any  $f \in A_m \times B_{n-1}$  must have  $f(n) = \iota$  and so  $A_m \times B_{n-1}$  is not tidy for  $\alpha$ . Therefore there is no tidy subgroup of  $G_1$  whose image in  $G_1/H_1$  is tidy.

EXAMPLE 6.6. Let  $G$  be as in Example 6.4. We describe certain subgroups of  $G$ . For each  $n \geq 0$ , define the subgroup  $P_n$  of  $\mathbb{Z}_2^{\mathbb{Z}}$  by

$$P_n = \{2^n \text{ periodic elements of } \mathbb{Z}_2^{\mathbb{Z}}\}.$$

Then  $P_0$  is the group of constant functions in  $\mathbb{Z}_2^{\mathbb{Z}}$ , so that  $|P_0| = 2$ , and in general  $P_n$  is a finite group with  $|P_n| = 2^{2^n}$ . Now recall that  $q: G \rightarrow \mathbb{Z}_2^{\mathbb{Z}}$  is a surjective homomorphism and define  $H_n = q^{-1}(P_n)$ ,  $n \geq 0$ . Then  $H_n$  is a closed discrete subgroup of  $G$  for each  $n$ .

It is easily verified that  $H_n \subset H_{n+1}$  for each  $n$  and that  $\bigcup_{n \geq 0} H_n$  is dense in  $G$ . Since each  $H_n$  is discrete we have  $s_{H_n}(\alpha|_{H_n}) = 1$  for each  $n$ . On the other hand, we have already seen in Example 6.4 that  $s_G(\alpha) = 2$ . Hence

$$\lim_{n \rightarrow \infty} s_{H_n}(\alpha|_{H_n}) \neq s_G(\alpha).$$

EXAMPLE 6.7. Let  $G$  be the compact group  $\mathbb{Z}_2^{\mathbb{Z}}$  with the product topology and let  $H$  be the two-element subgroup of  $G$  consisting of the constant functions. Let  $\alpha$  be the translation automorphism of  $G$ . Then  $\alpha(H) = H$ .

The subgroup  $\{e\}$  is tidy for  $\alpha|_H$ . However, the only tidy subgroup for  $\alpha$  is  $G$ . Hence there is no tidy subgroup of  $G$  whose intersection with  $H$  is  $\{e\}$ .

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